Homework 5:

Due: October 16, 2025 at 2:30p.m.

This homework must be typed in LaTeX and submitted via Gradescope.

Please ensure that your solutions are complete, concise, and communicated clearly. Use full sentences and plan your presentation before your write. Except where indicated, consider every problem as asking for a proof.

Problem 1. While playing with your pile of n rocks, you begin to wonder about the types of rocks you have. A type of rock is *overly common* if more than half of the rocks you have are of this type.

Unfortunately, you don't have the rock expertise to exactly tell what type a rock is, however, you do have access to a magic machine (i.e. a predicate) that is able to determine if two rocks share the same type.

(a) Describe an algorithm to determine if a collection of n rocks contains an overly common type of rock. If it does, return all of the overly common rocks. Your algorithm should make at most $O(n \log n)$ calls to the magic machine.

(b) Prove the correctness of your algorithm.

(c) Analyze the runtime of your algorithm, justifying that your algorithm makes at most $O(n \log n)$ calls to the machine.

[Hint: A top down divide and conquer algorithm has $O(\log n)$ runtime. A bottom up solution gives $O(n)$ runtime.]

Solution. □

Problem 2. Compute the product of the two polynomials using the Fast Fourier Transform (FFT):

- p(x) = 5x + 2,
- q(x) = 6x + 1.

Specify all (recursive) calls of the FFT algorithm as well as the outputs and the assignments of the temporary variables used during the execution.

Solution. □

Problem 3. Let S and T be two sets of integers in the range $[0, m]$ where m is a power of two. Use a single DFT to compute the following in $O(m \log_2 m)$ time:

- all elements contained in the set

$$S + T := \{s + t : s \in S, t \in T\}$$

- for each element $u \in S + T$, the number $ku := |\{(s, t) \in S \times T : s + t = u\}|$.

  [Hint: Find polynomials p_S and p_T of degree $< m$ that represents the sets S and T. Use x^z to represent item z.]